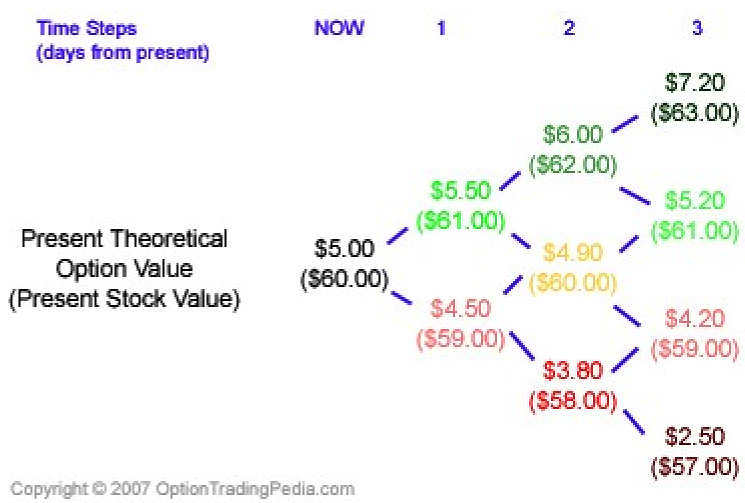
LECTURE 5

Chapter 4

BINOMIAL OPTION PRICING MODEL

Binomial Option Pricing Model (BOPM) was invented by Cox-Rubinstein in 1979. It was originally invented as a tool to explain the Black-Scholes Model to Cox's students. However, it soon became apparent that the binomial model is a more accurate pricing model for American Style Options. The binomial model is thus named as it returns 2 possibilities at any given time. Therefore, instead of assuming that an option trader will hold an option contract all the way to expiration like in the Black-Scholes Model, it calculates the value of that trader exercising that option contract with every possible future up and down moves on its underlying asset, reflecting its effects on the present value of that option, thus giving a more accurate theoretical price of an American Style option.

The binomial model produces a binomial distribution of all the possible paths that a stock price could take during the life of the option. A binomial distribution, or simply known as a "Binomial Tree", assumes that a stock can only increase or decrease in price all the way until the option expires and then maps it out in a "tree". Here is a simplified version of a binomial distribution just for illustration purpose:



It then fills in the theoretical value of that stock's options at each time step from the very bottom of the binomial tree all the way to the top where the final, present, theoretical value of a stock option is arrived. Any adjustments to stock prices at an ex-dividend date or option prices as a result of early exercise of American options are worked into the calculations at each specific time step .

Advantage of The Binomial Option Pricing Model

It can more accurately price American Style Options than the Black-Scholes Model as it takes into consideration the possibilities of early exercise and other factors like dividends.

Disadvantage of The Binomial Option Pricing Model

As it is much more complex than the Black-Scholes Model; it is slow and not useful for calculating thousands of option prices quickly.

Example:

BINOMIAL OPTION PRICING

Probability of direction of the stock up or down 50/50

Parameters	Current Stock Price	Probability (p)	Stock x p	Call Option Payoff if Exercised	Net after Repayment of Loan	Relationship between Payoff and Profit (leverage)	Value of the Call Option
Current Price=	$ 100.00						
Up probability (u) =		1.2	$ 120.00	$ 10.00	$ 30.00	3.0x	$ 6.06
Down probability (d) =		0.9	$ 90.00	$ -	$ -		
Range =			$ 30.00				

Exercise Call Option =	$ 110.00
Exercise time =	1 year
Borrowing Parameters	
Interest Rate =	10%
Borrowed Amount (P) per share=	$ 81.82
Interest Amount per share =	$ 8.18
Total	$ 90.00
Sources of Investment	
Loan	$ 81.82
Cash (Equity)	$ 18.18
Total Sources	$ 100.00

Fully Hedged Portfolio

Stock Price	90	120
Obligations for 3 Calls	0	-30
Payoff	90	90

HEDGE RATIO

Using the same example

One share =	3.0x	Calls
Current Price=	$ 100.00	
Value d x 100 =	$ 90.00	
value u x 100	$ 120.00	
Range	$ 30.00	
Cu=	$ 10.00	
Cd =	$ -	
The Ratio (Range / Call payoff)	1/3	

Hedge Ratio Formula:
(H) = (Cu - Cd) / (uSo - dSo)

Cu and Cd call option going up or down
uSo, dSo are the stock prices in the two states

uSo =	$ 120.00
dSo =	$ 90.00
Exercise Price =	$ 110.00
Cu =	$ 10.00
Cd =	$ -
Stock Price Range =	$ 30.00
Option Price Range =	$ 10.00
Hedge Ratio (H) =	**1/3**

Portfolio Hedging

Share per option	1/3
Written option would have an end-of-year value with certainty =	$ 30.00
PV=	$ 27.27
Set Value of the hedged position equal to the PV of certain payoff =	$ 33.33
Call's Value	$ 6.06

Testing a Mispriced option against the Hedge Ratio

	Assume Call option is mispriced at =	$6.50		
	Using the Hedge ratio, you get to the following strategy=	3.0x		
Sequence	**Strategy**	**Initial CF**	**At S1 = $ 90.00**	**At S1 = $ 120.00**
FIRST	Write 3 Calls at cost =	$ 19.50	$ -	$ (30.00)
SECOND	Buy one share =	$(100.00)	$ 90.00	$ 120.00
THIRD	Borrow the difference at 10% =	$ 80.50	$ (88.55)	$ (88.55)
	Total	**$ -**	**$ 1.45**	**$ 1.45**
	Present Value=	$ -	$ 1.32	$ 1.32
	Per option profit		$ 0.44	$ 0.44
			Note: Exactly the amount that the option is mispriced $6.50 - $6.06 =	$ 0.44

Example METHOD 1 - One Period Call Option Premium using BPM (page 113)
S=100
X=100
u= 1.25
d= .80
i= 0.07
C=?

Spot Price Path
Step 1: (S.u) – (S.d) = (100)(1.25) – (100)(.80) = 45

Call Price Path:
Step 2: (Su-X) – (Sd-X)>0 = 25 – 0 = 25
Step 3: h = Step 2 / Step 1 = 25/45 = 0.556
Step 4: PV(S.d) = 80 / (1+.07)^1 = 80/1.07 = 74.77
Step 5: S – PV(80) = 100 – 74.75 = 25.23
Step 6: Step 5 x Step 3 = (25.23)(0.556) = 14.02

Example METHOD 2 - One Period Call Option Premium using BPM (page 113)
Introducing p = [(1+i)-d]/(u-d)

S=100
X=100
u= 1.25
d= .80
i= 0.07
C=?

Spot Price Path:
Step 1: (S.u) – (S.d) = (100)(1.25) – (100)(.80) = 45

Call Price Path:
Step 2: (Su-X) – (Sd-X)>0 = 25 – 0 = 25
Step 3: h = Step 2 / Step 1 = 25/45 = 0.556
Step 4: The Value of p = (1+i-d)/(u-d) = (1.07 – 0.80) / (1.25-.80) = 0.6
Step 5: Then 1-p = 1-.60 = .40
Step 6: C = [p(Cu) + (1-p)(Cd)]/(1+i) = [(0.6)(25) + (0.4)(0)]/1.07= 14.02

Another Example:

S=60
X=50
u=1.15
d=.80
Rf or i=10%=0.1

Stock Price Path:
Su = (1.15)(60) = 69
Sd= (0.8)(60) = 48

Call Price Path
Cu = 69 – 50 = 19
Cd = 48 – 50 = 0 (if negative then 0)

To Price C:
h=(Cu-Cd)/(Su-Sd) = (19-0)/(69-48) = 19/21 = 0.9048 – basically buy 0.9048 shares for one call written.
Then

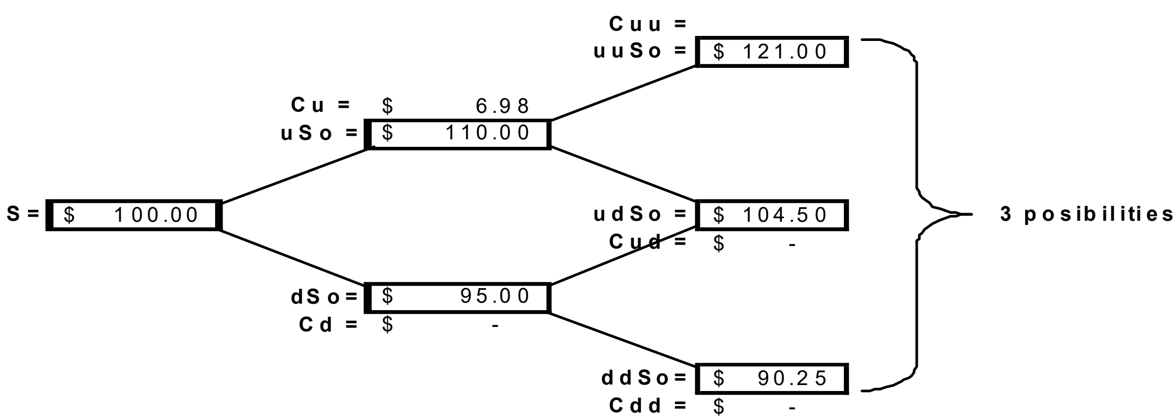
$(Su\ .h) - Cu = (0.9048)\ (69) - 19 = 43.43$
$(Sd.h) - Cd = (0.9048)\ (48) - 0 = 43.43$

Using $p = [(1\text{-}i) - d]/(u\text{-}d) = (1.1 - 0.8))/(1.15 - 0.8) = .857$

$C = [p\ (Cu) + (1\text{-}p)\ (Cd)]/(1+.i) = [(.857)\ (19) + (.143)\ (0)]/1.1 = \$14.80$

GENERALIZING THE TWO-STATE APPROACH

Scenarios	p	Increase / Decrease by
u =	1.100x	10.0%
d =	0.950x	-5.0%



H = (Cuu - Cud) / (uuSo - udSo)

A second period illustrated above will increase the number of possible outcomes at expiration, thus the model has three time points: today ($100), time 1: 110 or 95, time 2: $121, 104.50 or 90.25.

$C = [p\ (Cu + (1\text{-}p)Cd]/(1+i)$ – applying the second method for the two stage binomial pricing model:

$C = [(p^2Cu^2\ )+ 2p(1\text{-}p)Cud + (1\text{-}p)^2\ Cd^2]/(1+i)^2$

Example METHOD 2 -Two Period Call Option Premium using BPM (page 118)

S=100
X=100
u= 1.25
d= .80
i= 0.07
C=?
Step 1: $(S.u^2) - (S.d^2) = (100)(1.25)^2 - (100)(.80)^2 = 156.25 - 64 = 92.25$
Step 2: $(Su^2\text{-}X) - (Sd^2\text{-}X)>0 = 56.25 - 0 = 56.25$
Step 3: The Value of p = (1+i-d)/(u-d) = (1.07 – 0.80) / (1.25-.80) = 0.6
Step 4: Then 1-p = 1-.60 = .40
Step 5: Cu = [(0.6)(56.25) + (.40)(0)]/(1.07) = 31.54 and Cd = 0
Step 6: C = [(0.6)(31.54) + (.40)(0)]/(1.07) = 17.69

Another Example:

S=60
X=50
u=1.15
d=.80
Rf or i=10%=0.1

Stock Price Path – 3 possibilities:
$Su^2 = 60\ (1.15)^2 = 79.35$
Sud = 60 (1.15) (.80) = 55.20
$Sd^2 = 60\ (.80)^2 = 38.40$

Call Price Path:
$Cuu = Su^2 - X = 79.50 - 50 = 29.35$
Cud = Sud – X = 55.20 – 50 = 5.20
$Cu^2 = Sd^2 - X = 38.40 - 50 = 0$ (if negative put 0)

Using p = [(1-i) – d]/(u-d) = (1.1 – 0.8))/(1.15 – 0.8) = .857

Calculating C:
Cu = [p (Cuu) + (1-p) (Cud)]/(1+i) = [(0.857) (29.35) +(.143) (5.20)]/1.1=$23.54
Cd = [p (Cud) + (1-p) (Cdd)]/(1+i) = [(0.857) (5.20) +(0.143) (0)]/1.1 = $4.05

C = [p (Cu) + (1-p) (Cd)]/(1+i) = [(0.857) (23.54) +(0.143) (4.05)]/1.1 = $18.87

Binomial Option Pricing, Risk Premiums and Probabilities

So far the up and down limits are based on 50/50 probabilities respectively. Suppose we knew that the probability of an up move from the current stock price is 70% or 0.7 and the probability of the down movement is 30% or 0.3.. That mean, using the example above that for the first stage (0.7)(125)+(0.3)(80) = 111.50.

If the current stock price is $100, then investors must be discounting the expected stock price at a rate of 11.5 percent