6 DISCRETE-TIME OPTION PRICING: THE BINOMIAL MODEL

Until now, we have looked only at some basic principles of option pricing. Other than put–call parity, all we examined were rules and conditions, often suggesting limitations, on option prices. With put–call parity, we found that we could price a put or a call based on the prices of the combinations of instruments that make up the synthetic version of the instrument. If we wanted to determine a call price, we had to have a put; if we wanted to determine a put price, we had to have a call. What we need to be able to do is price a put or a call without the other instrument. In this section, we introduce a simple means of pricing an option. It may appear that we oversimplify the situation, but we shall remove the simplifying assumptions gradually, and eventually reach a more realistic scenario.

The approach we take here is called the **binomial model**. The word "binomial" refers to the fact that there are only two outcomes. In other words, we let the underlying price move to only one of two possible new prices. As noted, this framework oversimplifies things, but the model can eventually be extended to encompass all possible prices. In addition, we refer to the structure of this model as **discrete time**, which means that time moves in distinct increments. This is much like looking at a calendar and observing only the months, weeks, or days. Even at its smallest interval, we know that time moves forward at a rate faster than one day at a time. It moves in hours, minutes, seconds, and even fractions of seconds, and fractions of fractions of seconds. When we talk about time moving in the tiniest increments, we are talking about **continuous time**. We will see that the discrete time model can be extended to become a continuous time model. Although we present the continuous time model (Black–Scholes–Merton) in Section 7, we must point out that the binomial model has the advantage of allowing us to price American options. In addition, the binomial model is a simple model requiring a minimum of mathematics. Thus it is worthy of study in its own right.

6.1 The One-Period Binomial Model

We start off by having only one binomial period. This means that the underlying price starts off at a given level, then moves forward to a new price, at which time the option expires. Here we need to change our notation slightly from what we have been using previously. We let S be the current underlying price. One period later, it can move up to S^+ or down to S^-. Note that we are removing the time subscript, because it will not be necessary here. We let X be the exercise price of the option and r be the one period risk-free rate. The option is European style.

6.1.1 The Model

We start with a call option. If the underlying goes up to S^+, the call option will be worth c^+. If the underlying goes down to S^-, the option will be worth c^-. We know that if the option is expiring, its value will be the intrinsic value. Thus,

$$c^+ = \text{Max}(0,S^+ - X)$$
$$c^- = \text{Max}(0,S^- - X)$$

and put–call parity as

$$c_0 + X/(1 + r)^T = p_0 + [S_0 - PV(CF,0,T)]$$

which reflects the fact that, as we said, we simply reduce the underlying price by the present value of its cash flows over the life of the option.

5.8 The Effect of Interest Rates and Volatility

It is important to know that interest rates and volatility exert an influence on option prices. *When interest rates are higher, call option prices are higher and put option prices are lower.* This effect is not obvious and strains the intuition somewhat. When investors buy call options instead of the underlying, they are effectively buying an indirect leveraged position in the underlying. When interest rates are higher, buying the call instead of a direct leveraged position in the underlying is more attractive. Moreover, by using call options, investors save more money by not paying for the underlying until a later date. For put options, however, higher interest rates are disadvantageous. When interest rates are higher, investors lose more interest while waiting to sell the underlying when using puts. Thus, the opportunity cost of waiting is higher when interest rates are higher. Although these points may not seem completely clear, fortunately they are not critical. Except when the underlying is a bond or interest rate, interest rates do not have a very strong effect on option prices.

Volatility, however, has an extremely strong effect on option prices. *Higher volatility increases call and put option prices because it increases possible upside values and increases possible downside values of the underlying.* The upside effect helps calls and does not hurt puts. The downside effect does not hurt calls and helps puts. The reasons calls are not hurt on the downside and puts are not hurt on the upside is that when options are out-of-the-money, it does not matter if they end up more out-of-the-money. But when options are in-the-money, it does matter if they end up more in-the-money.

Volatility is a critical variable in pricing options. It is the only variable that affects option prices that is not directly observable either in the option contract or in the market. It must be estimated. We shall have more to say about volatility later in this reading.

5.9 Option Price Sensitivities

Later in this reading, we will study option price sensitivities in more detail. These sensitivity measures have Greek names:

- *Delta* is the sensitivity of the option price to a change in the price of the underlying.
- *Gamma* is a measure of how well the delta sensitivity measure will approximate the option price's response to a change in the price of the underlying.
- *Rho* is the sensitivity of the option price to the risk-free rate.
- *Theta* is the rate at which the time value decays as the option approaches expiration.
- *Vega* is the sensitivity of the option price to volatility.

EXHIBIT 15 One-Period Binomial Model

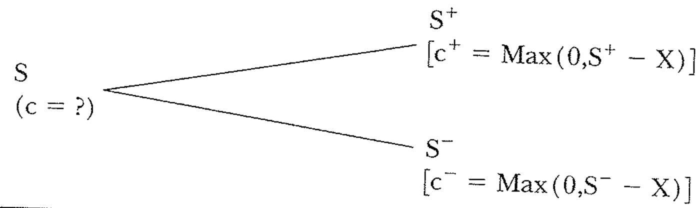


Exhibit 15 illustrates this scenario with a diagram commonly known as a **binomial tree**. Note how we indicate that the current option price, c, is unknown.

Now let us specify how the underlying moves. We identify a factor, u, as the up move on the underlying and d as the down move:

$$u = \frac{S^+}{S}$$

$$d = \frac{S^-}{S}$$

so that u and d represent 1 plus the rate of return if the underlying goes up and down, respectively. Thus, $S^+ = Su$ and $S^- = Sd$. To avoid an obvious arbitrage opportunity, we require that[21]

$$d < 1 + r < u$$

We are now ready to determine how to price the option. We assume that we have all information except for the current option price. In addition, we do not know in what direction the price of the underlying will move. We start by constructing an arbitrage portfolio consisting of one short call option. Let us now purchase an unspecified number of units of the underlying. Let that number be n. Although at the moment we do not know the value of n, we can figure it out quickly. We call this portfolio a hedge portfolio. In fact, n is sometimes called the **hedge ratio**. Its current value is H, where

$$H = nS - c$$

This specification reflects the fact that we own n units of the underlying worth S and we are short one call.[22] One period later, this portfolio value will go to either H^+ or H^-:

$$H^+ = nS^+ - c^+$$
$$H^- = nS^- - c^-$$

[21] This statement says that if the price of the underlying goes up, it must do so at a rate better than the risk-free rate. If it goes down, it must do so at a rate lower than the risk-free rate. If the underlying always does better than the risk-free rate, it would be possible to buy the underlying, financing it by borrowing at the risk-free rate, and be assured of earning a greater return from the underlying than the cost of borrowing. This would make it possible to generate an unlimited amount of money. If the underlying always does worse than the risk-free rate, one can buy the risk-free asset and finance it by shorting the underlying. This would also make it possible to earn an unlimited amount of money. Thus, the risky underlying asset cannot dominate or be dominated by the risk-free asset.

[22] Think of this specification as a plus sign indicating assets and a minus sign indicating liabilities.

Because we can choose the value of n, let us do so by setting H^+ equal to H^-. This specification means that regardless of which way the underlying moves, the portfolio value will be the same. Thus, the portfolio will be hedged. We do this by setting

$H^+ = H^-$, which means that

$nS^+ - c^+ = nS^- - c^-$

We then solve for n to obtain

$$n = \frac{c^+ - c^-}{S^+ - S^-} \tag{15}$$

Because the values on the right-hand side are known, we can easily set n according to this formula. If we do so, the portfolio will be hedged. A hedged portfolio should grow in value at the risk-free rate.

$H^+ = H(1 + r)$, or

$H^- = H(1 + r)$

We know that $H^+ = nS^+ - c^+$, $H^- = nS^- - c^-$, and $H = nS - c$. We know the values of n, S^+, S^-, c^+, and c^-, as well as r. We can substitute and solve either of the above for c to obtain

$$c = \frac{\pi c^+ + (1 - \pi)c^-}{1 + r} \tag{16}$$

where

$$\pi = \frac{1 + r - d}{u - d} \tag{17}$$

We see that the call price today, c, is a weighted average of the next two possible call prices, c^+ and c^-. The weights are π and $1 - \pi$. This weighted average is then discounted one period at the risk-free rate.

It might appear that π and $1 - \pi$ are probabilities of the up and down movements, but they are not. In fact, the probabilities of the up and down movements are not required. It is important to note, however, that π and $1 - \pi$ are the probabilities that would exist if investors were risk neutral. Risk-neutral investors value assets by computing the expected future value and discounting that value at the risk-free rate. Because we are discounting at the risk-free rate, it should be apparent that π and $1 - \pi$ would indeed be the probabilities if the investor were risk neutral. In fact, we shall refer to them as **risk-neutral probabilities**, and the process of valuing an option is often called **risk-neutral valuation.**[23]

[23] It may be helpful to contrast risk neutrality with risk aversion, which characterizes nearly all individuals. People who are risk neutral value an asset, such as an option or stock, by discounting the expected value at the risk-free rate. People who are risk averse discount the expected value at a higher rate, one that consists of the risk-free rate plus a risk premium. In the valuation of options, we are not making the assumption that people are risk neutral, but the fact that options can be valued by finding the expected value, using these special probabilities, and discounting at the risk-free rate creates the *appearance* that investors are assumed to be risk neutral. We emphasize the word "appearance," because no such assumption is being made. The terms "risk neutral probabilities" and "risk neutral valuation" are widely used in options valuation, although they give a misleading impression of the assumptions underlying the process.

6.1.2 *One-Period Binomial Example*

Suppose the underlying is a non-dividend-paying stock currently valued at \$50. It can either go up by 25 percent or go down by 20 percent. Thus, $u = 1.25$ and $d = 0.80$.

$$S^+ = Su = 50(1.25) = 62.50$$
$$S^- = Sd = 50(0.80) = 40$$

Assume that the call option has an exercise price of 50 and the risk-free rate is 7 percent. Thus, the option values one period later will be

$$c^+ = \text{Max}(0, S^+ - X) = \text{Max}(0, 62.50 - 50) = 12.50$$
$$c^- = \text{Max}(0, S^- - X) = \text{Max}(0, 40 - 50) = 0$$

Exhibit 16 depicts the situation.

EXHIBIT 16 One-Period Binomial Example

$S = 50$
$(c = ?)$

$S^+ = Su = 50(1.25) = 62.50$
$[c^+ = \text{Max}(0, S^+ - X)$
$= \text{Max}(0, 62.50 - 50) = 12.50]$

$S^- = Sd = 50(0.80) = 40$
$[c^- = \text{Max}(0, S^- - X)$
$= \text{Max}(0, 40 - 50) = 0]$

First we calculate π:

$$\pi = \frac{1 + r - d}{u - d} = \frac{1.07 - 0.80}{1.25 - 0.80} = 0.6$$

and, hence, $1 - \pi = 0.4$. Now, we can directly calculate the option price:

$$c = \frac{0.6(12.50) + 0.4(0)}{1.07} = 7.01$$

Thus, the option should sell for \$7.01.

6.1.3 *One-Period Binomial Arbitrage Opportunity*

Suppose the option is selling for \$8. If the option should be selling for \$7.01 and it is selling for \$8, it is overpriced—a clear case of price not equaling value. Investors would exploit this opportunity by selling the option and buying the underlying. The number of units of the underlying purchased for each option sold would be the value n:

$$n = \frac{c^+ - c^-}{S^+ - S^-} = \frac{12.50 - 0}{62.50 - 40} = 0.556$$

Thus, for every option sold, we would buy 0.556 units of the underlying. Suppose we sell 1,000 calls and buy 556 units of the underlying. Doing so would require an initial outlay of H = 556($50) − 1,000($8) = $19,800. One period later, the portfolio value will be either

$$H^+ = nS^+ - c^+ = 556(\$62.50) - 1{,}000(\$12.50) = \$22{,}250, \text{ or}$$
$$H^- = nS^- - c^- = 556(\$40) - 1{,}000(\$0) = \$22{,}240$$

These two values are not exactly the same, but the difference is due only to rounding the hedge ratio, n. We shall use the $22,250 value. If we invest $19,800 and end up with $22,250, the return is

$$\frac{\$22{,}250}{\$19{,}800} - 1 = 0.1237$$

that is, a risk-free return of more than 12 percent in contrast to the actual risk-free rate of 7 percent. Thus we could borrow $19,800 at 7 percent to finance the initial net cash outflow, capturing a risk-free profit of (0.1237 − 0.07) × $19,800 = $1,063 (to the nearest dollar) without any net investment of money. Other investors will recognize this opportunity and begin selling the option, which will drive down its price. When the option sells for $7.01, the initial outlay would be H = 556($50) − 1,000($7.01) = $20,790. The payoffs at expiration would still be $22,250. This transaction would generate a return of

$$\frac{\$22{,}250}{\$20{,}790} - 1 \approx 0.07$$

Thus, *when the option is trading at the price given by the model, a hedge portfolio would earn the risk-free rate*, which is appropriate because the portfolio would be risk free.

If the option sells for less than $7.01, investors would buy the option and sell short the underlying, which would generate cash up front. At expiration, the investor would have to pay back an amount less than 7 percent. All investors would perform this transaction, generating a demand for the option that would push its price back up to $7.01.

EXAMPLE 4

Consider a one-period binomial model in which the underlying is at 65 and can go up 30 percent or down 22 percent. The risk-free rate is 8 percent.

A. Determine the price of a European call option with exercise prices of 70.

B. Assume that the call is selling for 9 in the market. Demonstrate how to execute an arbitrage transaction and calculate the rate of return. Use 10,000 call options.

Solution to A: First find the underlying prices in the binomial tree. We have u = 1.30 and d = 1 − 0.22 = 0.78.

$$S^+ = Su = 65(1.30) = 84.50$$
$$S^- = Sd = 65(0.78) = 50.70$$

Then find the option values at expiration:

$$c^+ = \text{Max}(0{,}84.50 - 70) = 14.50$$
$$c^- = \text{Max}(0{,}50.70 - 70) = 0$$

The risk-neutral probability is

$$\pi = \frac{1.08 - 0.78}{1.30 - 0.78} = 0.5769$$

and $1 - \pi = 0.4231$. The call's price today is

$$c = \frac{0.5769(14.50) + 0.4231(0)}{1.08} = 7.75$$

Solution to B: We need the value of n for calls:

$$n = \frac{c^+ - c^-}{S^+ - S^-} = \frac{14.50 - 0}{84.50 - 50.70} = 0.4290$$

The call is overpriced, so we should sell 10,000 call options and buy 4,290 units of the underlying.

Sell 10,000 calls at 9	+90,000
Buy 4,290 units of the underlying at 65	−278,850
Net cash flow	−188,850

So we invest 188,850. The value of this combination at expiration will be

If $S_T = 84.50$,

$$4{,}290(84.50) - 10{,}000(14.50) = 217{,}505$$

If $S_T = 50.70$,

$$4{,}290(50.70) - 10{,}000(0) = 217{,}503$$

These values differ by only a rounding error.
The rate of return is

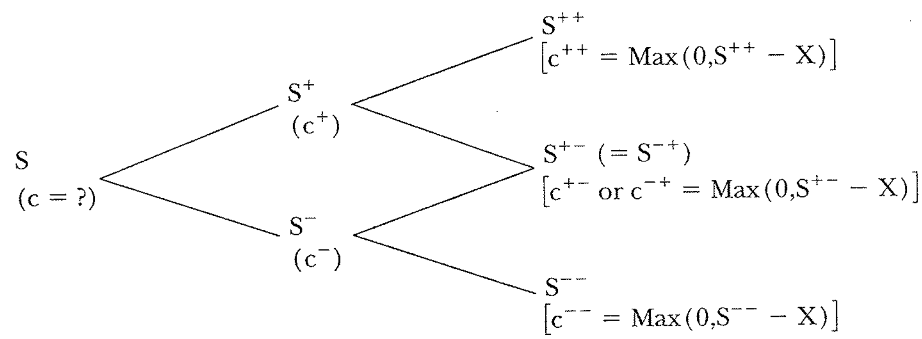
$$\frac{217{,}505}{188{,}850} - 1 = 0.1517$$

Thus, we receive a risk-free return almost twice the risk-free rate. We could borrow the initial outlay of \$188,850 at the risk-free rate and capture a risk-free profit without any net investment of money.

6.2 The Two-Period Binomial Model

In the example above, the movements in the underlying were depicted over one period, and there were only two outcomes. We can extend the model and obtain more-realistic results with more than two outcomes. Exhibit 17 shows how to do so with a two-period binomial tree.

EXHIBIT 17 Two-Period Binomial Model



In the first period, we let the underlying price move from S to S^+ or S^- in the manner we did in the one-period model. That is, if u is the up factor and d is the down factor,

$$S^+ = Su$$
$$S^- = Sd$$

Then, with the underlying at S^+ after one period, it can either move up to S^{++} or down to S^{+-}. Thus,

$$S^{++} = S^+u$$
$$S^{+-} = S^+d$$

If the underlying is at S^- after one period, it can either move up to S^{-+} or down to S^{--}.

$$S^{-+} = S^-u$$
$$S^{--} = S^-d$$

We now have three unique final outcomes instead of two. Actually, we have four final outcomes, but S^{+-} is the same as S^{-+}. We can relate the three final outcomes to the starting price in the following manner:

$$S^{++} = S^+u = Suu = Su^2$$
$$S^{+-} \text{ (or } S^{-+}) = S^+d \text{ (or } S^-u) = Sud \text{ (or } Sdu)$$
$$S^{--} = S^-d = Sdd = Sd^2$$

Now we move forward to the end of the first period. Suppose we are at the point where the underlying price is S^+. Note that now we are back into the one-period model we previously derived. There is one period to go and two outcomes. The call price is c^+ and can go up to c^{++} or down to c^{+-}. Using what we know from the one-period model, the call price must be

$$c^+ = \frac{\pi c^{++} + (1 - \pi)c^{+-}}{1 + r} \quad \textbf{(18)}$$